Exercise 4E

1 a $\int \text{arsinh}\,x\,\text{d}x$

Let $u = \text{arsinh}\,x$

$\sinh u = x$

$\cosh u \dfrac{\text{d}u}{\text{d}x} = 1$

$$\begin{aligned}\frac{\text{d}u}{\text{d}x} &= \frac{1}{\cosh u}\\ &= \frac{1}{\sqrt{\sinh^2 u + 1}}\\ &= \frac{1}{\sqrt{x^2+1}}\end{aligned}$$

Let $\dfrac{\text{d}v}{\text{d}x} = 1 \Rightarrow v = x$

$$\begin{aligned}\int \text{arsinh}\,x\,\text{d}x &= x\,\text{arsinh}\,x - \int \frac{x}{\sqrt{x^2+1}}\,\text{d}x\\ &= x\,\text{arsinh}\,x - \sqrt{x^2+1} + c \text{ as required}\end{aligned}$$

b $$\begin{aligned}\int_0^1 \text{arsinh}\,x\,\text{d}x &= \left[x\,\text{arsinh}\,x - \sqrt{x^2+1}\right]_0^1\\ &= \left(\text{arsinh}\,(1) - \sqrt{1^2+1}\right) - (-1)\\ &= \ln\left(1+\sqrt{2}\right) - \sqrt{2} + 1\\ &= 0.467 \text{ (3 s.f.)}\end{aligned}$$

c From part **a**

$\int \text{arsinh}\,x\,\text{d}x = x\,\text{arsinh}\,x - \sqrt{x^2+1} + c$

Let $u = 2x+1 \Rightarrow \dfrac{\text{d}u}{\text{d}x} = 2 \Rightarrow \text{d}x = \dfrac{1}{2}\text{d}u$

$$\begin{aligned}\int \text{arsinh}\,(2x+1)\,\text{d}x &= \frac{1}{2}\int \text{arsinh}\,u\,\text{d}u\\ &= \frac{1}{2}\left(u\,\text{arsinh}\,u - \sqrt{u^2+1}\right) + c\\ &= \frac{1}{2}\left((2x+1)\,\text{arsinh}\,(2x+1) - \sqrt{4x^2+4x+2}\right) + c\end{aligned}$$

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2 $\int \arctan 3x \, dx$

Let $u = \arctan 3x$

$\tan u = 3x$

$\sec^2 u \frac{du}{dx} = 3$

$$\frac{du}{dx} = \frac{3}{\sec^2 u}$$
$$= \frac{3}{1 + \tan^2 u}$$
$$= \frac{3}{1 + 9x^2}$$

Let $\frac{dv}{dx} = 1 \Rightarrow v = x$

$$\int \arctan 3x \, dx = x \arctan 3x - \int \frac{3x}{1 + 9x^2} dx$$
$$= x \arctan 3x - \frac{1}{6}\int \frac{18x}{1 + 9x^2} dx$$
$$= x \arctan 3x - \frac{1}{6}\ln\left(1 + 9x^2\right) + c \text{ as required}$$

3 a $\int \text{arcosh}\, x \, dx$

Let $u = \text{arcosh}\, x$

$\cosh u = x$

$\sinh u \frac{du}{dx} = 1$

$$\frac{du}{dx} = \frac{1}{\sinh u}$$
$$= \frac{1}{\sqrt{\cosh^2 u - 1}}$$
$$= \frac{1}{\sqrt{x^2 - 1}}$$

Let $\frac{dv}{dx} = 1 \Rightarrow v = x$

$$\int \text{arcosh}\, x \, dx = x \,\text{arcosh}\, x - \int \frac{x}{\sqrt{x^2 - 1}} dx$$
$$= x \,\text{arcosh}\, x - \sqrt{x^2 - 1} + c \text{ as required}$$

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3 b $\int_1^2 \operatorname{arcosh} x \, dx = \left[x \operatorname{arcosh} x - \sqrt{x^2-1}\right]_1^2$

$= \left[x \ln\left(x+\sqrt{x^2-1}\right) - \sqrt{x^2-1}\right]_1^2$

$= \left(2\ln\left(2+\sqrt{2^2-1}\right) - \sqrt{2^2-1}\right) - \left(\ln\left(1+\sqrt{1^2-1}\right) - \sqrt{1^2-1}\right)$

$= \left(2\ln\left(2+\sqrt{3}\right) - \sqrt{3}\right) - \ln(1)$

$= \ln\left(2+\sqrt{3}\right)^2 - \sqrt{3}$

$= \ln\left(7+4\sqrt{3}\right) - \sqrt{3}$ as required

4 a $\int \arctan x \, dx$

Let $u = \arctan x$

$\tan u = x$

$\sec^2 u \frac{du}{dx} = 1$

$\frac{du}{dx} = \frac{1}{\sec^2 u}$

$= \frac{1}{1+\tan^2 u}$

$= \frac{1}{1+x^2}$

Let $\frac{dv}{dx} = 1 \Rightarrow v = x$

$\int \arctan x \, dx = x \arctan x - \int \frac{x}{1+x^2} dx$

$= x \arctan x - \frac{1}{2}\ln\left(1+x^2\right) + c$ as required

b $\int_{-1}^{\sqrt{3}} \arctan x \, dx = \left[x \arctan x - \frac{1}{2}\ln\left(1+x^2\right)\right]_{-1}^{\sqrt{3}}$

$= \left(\sqrt{3}\arctan\sqrt{3} - \frac{1}{2}\ln(1+3)\right) - \left(-\arctan(-1) - \frac{1}{2}\ln\left(1+(-1)^2\right)\right)$

$= \left(\frac{\pi\sqrt{3}}{3} - \frac{1}{2}\ln 4\right) - \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right)$

$= \frac{\pi\sqrt{3}}{3} - \frac{1}{2}\ln 4 - \frac{\pi}{4} + \frac{1}{2}\ln 2$

$= \frac{4\pi\sqrt{3}-3\pi}{12} - \frac{1}{2}\ln\left(\frac{4}{2}\right)$

$= \frac{\pi\left(4\sqrt{3}-3\right)}{12} - \frac{1}{2}\ln 2$ as required

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4 c $y = \arctan x \, dx$

$$R = \pi \times 3 - 2\int_0^3 \arctan x \, dx$$

$$= 3\pi - 2\left[x \arctan x - \frac{1}{2}\ln\left(1+x^2\right)\right]_0^3$$

$$= 3\pi - 2\left[\left(3\arctan 3 - \frac{1}{2}\ln\left(1+3^2\right)\right) - \left(-\frac{1}{2}\ln(1)\right)\right]$$

$$= 3\pi - 2\left(3\arctan 3 - \frac{1}{2}\ln 10\right)$$

$$= 3\pi - 6\arctan 3 + \ln 10$$

$$= 4.23 \text{ (3 s.f.)}$$

5 a $\displaystyle\int_0^{\frac{\sqrt{2}}{2}} \arcsin x \, dx$

Let $u = \arcsin x$

$\sin u = x$

$$\cos u \frac{du}{dx} = 1$$

$$\frac{du}{dx} = \frac{1}{\cos u}$$

$$= \frac{1}{\sqrt{1-\sin^2 u}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

Let $\dfrac{dv}{dx} = 1 \Rightarrow v = x$

$$\int_0^{\frac{\sqrt{2}}{2}} \arcsin x \, dx = \left[x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx\right]_0^{\frac{\sqrt{2}}{2}}$$

$$= \left[x \arcsin x + \sqrt{1-x^2}\right]_0^{\frac{\sqrt{2}}{2}}$$

$$= \left(\frac{\sqrt{2}}{2}\arcsin\left(\frac{\sqrt{2}}{2}\right) + \sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}\right) - 1$$

$$= \left(\frac{\sqrt{2}}{2} \times \frac{\pi}{4} + \sqrt{1-\left(\frac{2}{4}\right)}\right) - 1$$

$$= \frac{\pi\sqrt{2}}{8} + \frac{\sqrt{2}}{2} - 1$$

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5 b $\int_0^1 x\arctan x\,dx$

Let $u = \arctan x$

$\tan u = x$

$\sec^2 u\frac{du}{dx} = 1$

$\frac{du}{dx} = \frac{1}{\sec^2 u}$

$= \frac{1}{1+\tan^2 u}$

$= \frac{1}{1+x^2}$

Let $\frac{dv}{dx} = x \Rightarrow v = \frac{1}{2}x^2$

$\int_0^1 x\arctan x\,dx = \left[\frac{1}{2}x^2\arctan x - \frac{1}{2}\int\frac{x^2}{1+x^2}\,dx\right]_0^1$

$\frac{x^2}{1+x^2} = \frac{1+x^2-1}{1+x^2}$

$= \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}$

$= 1 - \frac{1}{1+x^2}$

Therefore:

$\int_0^1 x\arctan x\,dx = \frac{1}{2}\left[x^2\arctan x - \int\left(1-\frac{1}{1+x^2}\right)dx\right]_0^1$

$= \frac{1}{2}\left[x^2\arctan x - x + \arctan x\right]_0^1$

$= \frac{1}{2}(\arctan 1 - 1 + \arctan 1)$

$= \arctan 1 - \frac{1}{2}$

$= \frac{\pi}{4} - \frac{1}{2}$

$= \frac{\pi - 2}{4}$

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6 $\int \operatorname{arcsec} x \, dx$

Let $u = \operatorname{arcsec} x \Rightarrow \frac{du}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$

Let $\frac{dv}{dx} = 1 \Rightarrow v = x$

$$\begin{aligned}\int \operatorname{arcsec} x \, dx &= x \operatorname{arcsec} x - \int \frac{x}{x\sqrt{x^2 - 1}} dx \\ &= x \operatorname{arcsec} x - \int \frac{1}{\sqrt{x^2 - 1}} dx \\ &= x \operatorname{arcsec} x - \operatorname{arcosh} x + c \\ &= x \operatorname{arcsec} x - \ln\left(x + \sqrt{x^2 - 1}\right) + c\end{aligned}$$

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